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GEOMETRY ASSOCIATED WITH THE GENERALIZATION OF CONVEXITY

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ABSTRACT

In this paper we have discussed the generalization of convexity in a space and noted some interesting geometry arising from it. The underlying space is $\mathbf{R}^d$, d >1, is a d- dimensional Euclidean space. Convexity of a set of points in $\mathbf{R}^d$ is introduced and an operator on such sets namely the convex hull is also defined. Under this operator convexhull, we formulate some propositions. Since, convexity is intimately related to the connectedness of the space , generalizations to lines and planes brings to fore some topological restrictions. The study gains importance due to the global nature of the problem. Richard Polleck, Raphael Wenger and others investigated this problem, we refer to [1][2][3] [4] Jacob E. Goodman extends this study owing to its relationship with the discrete geometric nature of the set[1]. Section 2 has some basic concepts from geometry and analysis. Section 3 deals with propositions and scope of the problem.

KEYWORDS: Geometry, Convexity, Dimensional Euclidean Space

INTRODUCTION

Convex Sets of $\mathbf{R}^d$ and Linear Translates

Let $\mathbf{R}^d$, d> 1 be the d-dimensional Euclidean space and the topology is the induced topology arising from the Euclidean metric.Let S $\subset \mathbf{R}^d$, either finite or locally finite. If S is locally finite then every point of S has a neighbourhood which has finite number of points. In any case we would expect S to be discrete.

A linear transformation T: $\mathbf{R}^d \rightarrow \mathbf{R}^k$ is a map which is completely determined by its action on basis elements. In geometry we are trying to get a linear translate of the subspaces of $\mathbf{R}^d$ and quotienting $\mathbf{R}^d$ over such subspaces. We will explain this with some details.

If v is vector space over H and W$\subset$ V, is a subspace over H, where V and W are finite dimensional. Then V/W the quotient space where the elements are cosets is also a vector space over H. The members of V/W are the cosets. This is clear from linear algebra of vector spaces.

Definition 2.1 Let $S\subset\mathbf{R}^d$,of points in $\mathbf{R}^d$. S is said to be convex if the line segment joining any two points of S lies completely in S . For example , If S is a line segment then S is convex. A line segment is defined by the following property. If x,y $\in$ S and 0< λ <1, λx+(1-λ)y $\in$ S.

Isf S is such that any two points of S joined by the line segment, lies in S then S is convex. See figure (i) and (ii) for S

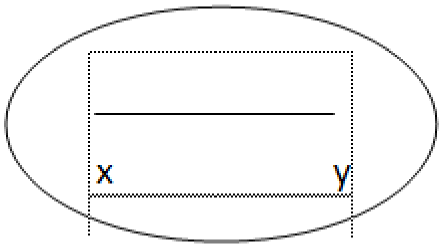

Figure (i)

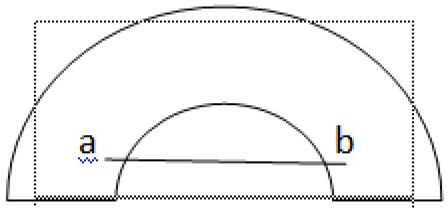

Figure (ii)

Basic Properties of a Convex Set

Let H be any set associated with S is the set called convex hull of S, denoted by ConvS, which contains S.

If S is a set of four points then its convex hull is as shown in the figure (see figure (iii)).

S = {a,b,c,d}

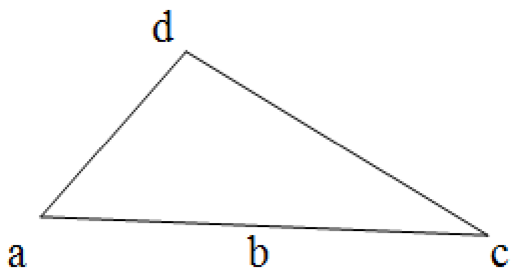

Figure (iii)

Convex hull of four points figure (iii)

Clearly Conv S Satisfies the following Property

- For the sets S and T, S $\subset$ T , then ConvS $\subset$ ConvT

  Convex hull satisfies the monotonicity property

- Conv(ConvS)=ConvS

  This property is the idempotency property.

- Let Conv S=S and x,y$\notin$ S, further x$\neq$ y ie x and y are different. Then y$\in$ Conv(S U{y}) implies x∉Conv(SU {y}).

  The property 3 is called anti-exchange property of the convex hull operator.

Observe that , property 3 induces a partial order on the sets of $\mathbb{R}^d$. The following proposition implies this property of anti-exchange.

Proposition 2.3 Let S $\subset\mathbb{R}^d$ be convex and ConvS denote its convex hull , then S^1, the complete of S is partially ordered under anti-exchange property of convex hull.

Proof: Suppose the point set of $\mathbb{R}^d$ be convex and con V S denote its convex hull such that con V S= S define partial order in S', the compliment of S by the anti-exchange property i.e , x, y $\in S'$, x $\leq$ y, whenever $x \neq$ y, and x$\in$ con V (S U$\{y\}$) $\Rightarrow y \notin$ con V (S U$\{x\}$)

Clearly x $\leq$ x

Suppose x $\leq$ y and y $\leq$ x, x, y $\in S'$, $x \neq$ y , then x = y

$\neq$x$\in$ con V (S U$\{y\}$) $\Rightarrow y \notin$ con V (S U$\{x\}$)

For y $\leq$ x implies

y$\in$ con V (S U$\{x\}$) x$\notin$ con V (S U$\{y\}$)

Together we infer that x = y

Anti-exchange property is clearly transititve. x $\leq$ y and y $\leq$ z for all x, y ,z $\in$ S x $\neq$ y ,

y$\neq$ z $\Rightarrow$ x $\leq$ z .

Thus , anti-exchange property on s' partitions S' into equivalence classes

Remark:1.2.3 The property antiexchange gives us an idea that how far away are the point from a convex set.

- The three properties defined above are generally called the convex hull operations and gives the convexity structure.

Definition 1.2.4 An affine invariance of the convexity structure is that convex hull operating commuter with the action of the affine group.

Let A(d , $\mathbb{R}^{\dagger}$) = $\left\{ \begin{pmatrix} l' & \alpha \\ 0 & 1 \end{pmatrix}\right.$: l$\in$ GL(d, $\mathbb{R}^{\dagger}$), $\alpha \in \mathbb{R}^{\dagger}d$ } andbe the group of non-singular transformations with translations. If σ $\in$ A (d , $\mathbb{R}^{\dagger}$) and S $\subset \mathbb{R}^d$ .Then S is said to be convex if

Con V S= S

Under σ : S $\to$ S'

If x $\in$ conV S then σ $\in$ con V S'

In other words $\mathbb{R}^d$**admits convexity structure** naturally.

Convexity Structure for Line Sets and in General K-Flats

In this section we generalize the notion of convexity for sets of $\mathbb{R}^d$, which are not point sets but they are line sets or more generally k=flats. This leads to what is known as Geometric transversal theory. From differential geometric point of view there are examples , which provide us the notion of transversality on surfaces , and in manifolds of higher dimensions we give two examples in each of these cases.

Following theorem due to [3] provides the convexity structure for $G_{K,.d}$ are Grassmanians.

Theorem1.3.1 There is no notion of convexity for lines or higher dimensional flats that is non-singular –affine invariant, satisfying anti-exchange property in which all convex sets are connected.

For convexity to hold for the line (or K- flat) sets , we have to give up the connectedness. Thus , if we assume that a convex set of flats are not always connected then we notice a rich theory , extending the notion of convexity and properties associated with the point sets.

Remark 1.3.1

If the space is one-dimensional then the points would sit on the line. if the space is two dimensional . then the points and lines occur in this two dimensional setting. Points lying both on lines and plane itself. Thus, points and lines are endowed with the property 'surrounded by' .This property of being 'surrounded by' is a key to generalize the convexity structure to the sets whose points are lines and K-flats in $\mathbb{R}^d$.

In the context of a flat F the notion of convex hull is defined by taking a re-look into the point sets of $\mathbb{R}^d$.

Let F denote a K- flat , then a hyper-plane will be a flat of co=dimension 1.

Definition 1.3.2 : Let x be a point in a flat F . Then we say that the point x in F is surrounded by a set of points S in F if any hyper-plane H in F passing through x lies strictly between two parallel hyper-planes H_1 and H_2 in F each containing a point of S

The following figure is indicative and the description clearly suggests that such a hyper-plane is trapped by points of S in case it tries to escape by a continuous translation to infinity

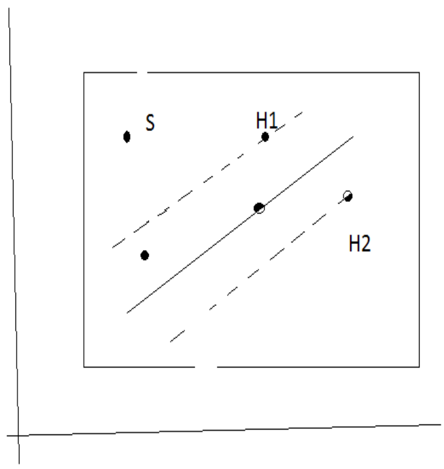

Figure (iv)

From this definition the following properties are hold good

- x $\in$ conv S if and only if there is a flat F containing x with which x is surrounded by the points of S lying in F.
- Sometimes S may be a lower dimensional set and therefore by saying x is surrounded by the points of S instead of it being surrounded by the points of S lying in F may not hold.
- x $\in$ conv S if every convex point set meeting every point of S also meets x. This is quite evident as it amounts to

say that conv S is the intersection of all the convex point sets containing S.

Further 'surrounded by' makes a perfect meaning when the basic objects are flats of fixed dimension k , k =1,2,3------ rather than simply points. Further the properties 1 and 2 are valid in that setting as equivalent statements. Moreover they imply the basic property of convex structure associated with the convex hull operations in $\mathbf{R}^d$. This leads to define the convex hull of k-flats

Definition 1.3.3: Let $\mathcal{E}$ denote a set of k-flats in $\mathbf{R}^d$. Conv $\mathcal{E}$ denote its convex hull. A

k-flat $\mathcal{E}'$ belongs to Conv $\mathcal{E}$ in $\mathbf{R}^d$ if it satisfies either of the following conditions

- There is a flat F containing $\mathcal{E}'$ within which $\mathcal{E}$ ' is surrounded by the flat of $\mathcal{E}$ lying in F
- Every convex point set meeting all the members of $\mathcal{E}$ also meets $\mathcal{E}'$.

CONCLUSIONS

These definitions will enable us to provide examples for Geometric transversal theory. We have considered a discretized version of a manifold and the subsets of them is studied for the convexity structure.

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