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G*- COMPACT SPACE

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ABSTRACT

In this paper g*- isolated point, g*- compact, g*-locally compact, g*-sequencially compact, g*-countably compact are introduced and the relationship between these concepts are studied.

KEYWORDS: G*-Isolated Point, G*-Compact, G*-Locally Compact, G*-Sequencially Compact, G*-Countably Compact

1. INTRODUCTION

Levine **[1]** introduced the class of g-closed sets in 1970 and M.K.R.S.Veerakumar **[5]** introduced g*-closed sets in 1991. In this paper g*- compact spaces, g*-locally compact spaces, g*-sequencially compact spaces, g*-countably compact spaces are defined and their properties are investigated.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space (X, τ) is called

1) *generalized closed*set (briefly *g-closed*)**[1]** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

2) *generalized star closed*set (briefly *g*-closed*)**[5]** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in (X, τ).

Definition 2.2[5]: A function f : (X,τ) → (Y,σ) is called

1) g^*-irresolute if f^{-1} (V) is a g^*-closed set of (X,τ) for every g^*-closed set V of (Y,σ).

2) g^*-continuous if f^{-1} (V) is a g^*-closed set of (X,τ) for every closed set V of (Y,σ).

3) strongly g^*-continuous if f^{-1}(V) is a closed set of (X,τ) for every g*-closed set V of (Y,σ).

4) g^*-resoluteif f (V) is g^*-closed in Y whenever V is g^*-closed in X.

Definition 2.3[3]: Let (X,τ) be a topological space and $x \in X$. Every *g*-open* set containing x is said to be a *g*-neighbourhood* of x.

Definition 2.4[3]: Let Abe a subset of X. A point $x \in X$ is said to be a *g*-limit point* of A if every *g*-neighbourhood* of x contains a point of A other than x.

Definition 2.5[3]: Let Abe a subset of a topological space (X,τ). $g^*cl(A)$ is defined to be the intersection of all g^*-*closed* sets containing A.

Note: g^*cl(A) need not be g^*-closed, since intersection of g^*-closed sets need not be

g^*-closed. But if A is g^*-closed then g^*cl(A) = A.

Definition 2.6[3]: The topological space (X,τ) is said to be *g*-multiplicative* if arbitrary intersection of g^*-closed sets is g^*-closed. Equivalently arbitrary union of g^*-open sets is g^*-open.

Note: If (X,τ) is g^*-multiplicative then A = g^*cl (A) if and only if A is g^*-closed.

Definition 2.7[2]: A collection C of subsets of X is said to have the finite intersection property if fore very sub collection $\{C_1, C_{2,\ldots\ldots} C_n\}$ of C then the intersection $C_1 \cap C_2 \cap \ldots\ldots \cap C_n$ is not empty.

Definition 2.8[4]: A topological space (X,τ) is said to be g^*T_2 space if for every pair of distinct points x, y in X, there exists disjoint g*-open sets U and V in X such that $x \in U$ and $y \in V$.

3. G^*- COMPACT SPACE

Definition 3.1: A collection $\{U_\alpha\}_{\alpha\in\Delta}$ of g*-open sets in X is said to be g^*-open cover of X if X = $\bigcup_{\alpha\in\Delta} U_\alpha$.

Definition 3.2: A topological space (X,τ) is said to be g*-compact if every g*-open covering of X contains a finite sub collection that also covers X. A subset A of X is said to be g*-compact if every g*-open covering of A contains a finite sub collection that also covers A.

Remark 3.3: A Topological Space (X,τ) Is

(1) g*-compactness ⇒ compactness

(2) Any finite space is g^*-compact.

Example 3.4: Let (X,τ) be infinite cofinite topological space. Then G*O(X) = $\{\Phi, X, A /$ A^c is finite$\}$ =G*O(X). Let $\{U_\alpha\}_{\alpha\in\Delta}$be an arbitrary g*-open cover for X. Let U_{α_0}be one g*-open set in the open cover $\{U_\alpha\}_{\alpha\in\Delta}$. Then X $-U_{\alpha_0}$is finite, say$\{x_1, x_2, x_3, \ldots\ldots x_n\}$.Choose U_{α_i} such that $x_{\alpha_i} \in U_{\alpha_i}$ for i = 1,2,.........n. Then X= $U_{\alpha_0} \cup U_{\alpha_1} \cup \ldots\ldots \cup U_{\alpha_n}$. The spaceis g*-compact and hence compact

Theorem 3.5: A g*-closed subset of g*-compact space is g*-compact.

Proof : Let A be a g*-closed subset of a g*-compact space (X,τ) and $\{U_\alpha\}_{\alpha\in\Delta}$ be a g*-open cover for A. Then $\{ \{U_\alpha\}_{\alpha\in\Delta}, (X - A) \}$ is a g*-open cover for X.SinceXis g^*-compact, there exists $\alpha_1, \alpha_{2,\ldots\ldots\ldots}\alpha_n \in \Delta$ such that X= $U_{\alpha_1} \cup \ldots\ldots \cup U_{\alpha_n}$ which proves A is g^*-compact.

Remark 3.6: The converse of the above theorem need not be true as seen in the following example.

Example 3.7: A set which is g^*-compact but not g^*-closed.Let X={ a, b, c},τ ={Φ,{a},{a,b},X}. Then (X,τ) is g^*-compact. The subset A = {b} is g^*-compact but not g^*-closed

Theorem 3.8[4]: Let (X,τ) be a g*- multiplicative g^*T_2space. Then every g*- compact subset of X is g*- closed.

Proof: Let Y be a g*- compact subset of g* T_2- space. Let $x_0 \in X - Y$. For each point y ∈ Y. There exists disjoint g*- open sets U_y and V_y containing y and x_0 respectively. Therefore $\{U_y / y\in Y\}$ is a g*-open cover for Y. Now there exists $\{y_1, y_{2,\ldots\ldots\ldots} y_n\} \in Y$ such that Y $\subseteq \bigcup_{i=1}^{n} U_{y_i}$ = U (say). Let V = $\bigcap_{i=1}^{n} V_{y_i}$. Then V is g*-open. Since X is g*-multiplicative, U is g*-open. Obviously $U \cap V = \varphi$. Therefore V is a g*-neighbourhood of x_o contained in X−Y. Therefore X−Y is g*-open and hence Y is g *-closed.

Note: The converse of theorem 3.8 is true if (X,τ) is g*-multiplicative and g*-T_2.

Remark 3.9: In theorem 3.8, the condition is necessary. An infinite cofinite topological space is g* -multiplicative but not g^*T_2. In this space all subsets are g*-compact but only finite sets are g*-closed.

Theorem 3.10: Let Y be a g*-compact subset of a g*-T_2 space X and $x_0 \notin Y$. Then

there exists disjoint g*-open sets U and V of X containing x_0 and Y respectively.

Proof: Theg*-open setsU and Vdiscussed in th**e** proofoftheorem3.8 are disjoint g*-open sets containing Y and x_0 respectively.

Theorem 3.11: Let (X,τ) and (Y,σ) be two topological spaces and $f:(X,\tau)\to(Y,\sigma)$ be a function. Then

1. f is g*-irresolute and A is a g*-compact subset of X $\Rightarrow f$ (A) is a g*-compact subset of Y.

2. f is one to one, g*-resolute and B is a g*-compact subset of Y $\Rightarrow f^{-1}$ (B) is a g*-compact subset of X.

3. f is g*-resolute, X is g*-compact, Y is g*-multiplicative and $g^*T_2 \Rightarrow f$ is a g*-resolute function.

4. f is g*-resolute and Y is g*-compact and X is g*-multiplicative and $g^*T_2 \Rightarrow f$ is a g*-irresolute function.

Proof: (1) & (2) Obviously from the definitions. (3) Proof follows from (1) and theorem (3.8). (4) Proof follows from (2) and theorem (3.8).

Definition 3.12[2]: A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite sub collection $\{C_1, C_2, \ldots C_n\}$ of $\mathcal{C}$ then $C_1\cap C_2\cap\ldots\ldots\cap C_n \neq \Phi$.

Theorem 3.13: A topological space (X,τ) is g*-compact if and only if for every collection $\mathcal{C}$of g^*-closed sets in X having the finite intersection property, the intersection $\bigcap_{c\in C} C$ of all elements of $\mathcal{C}$ is non-empty.

Proof: Let (X,τ) be g^*-compact and $\mathcal{C}$be a collection of g^*-closed sets with finite intersection property.Toprove $\bigcap_{c\in C} C$ =Φ.Suppose $\bigcap_{c\in C} C$ =Φthen $\bigcap_{c\in C}(X-C)$ =X. Therefore $\{X-C \;/\; c\in C\}$ is a g*-open cover for X.Then there exists $C_1, C_2, \ldots C_n \in \mathcal{C}$such that $\bigcup_{i=1}^{n}(X-C_i)$ =X. Therefore $\bigcap_{i=1}^{n} C_i = \Phi$ which is acontradiction. Therefore $\bigcap_{c\in C} C \neq \Phi$ .Conversely, assume the hypothesis given in the statement. To prove X is g^*-compact. Let $\{U_\alpha\}\,\alpha\varepsilon\Delta$ be a g*-open cover for X.Then $\bigcup_{\alpha\varepsilon\Delta} U_\alpha$ = X $\Rightarrow \bigcap_{\alpha\varepsilon\Delta}(X-U_\alpha) = \Phi$ .By the hypothesisthere exists $\alpha_1, \alpha_2 \ldots\ldots\ldots \alpha_n$ such that $\bigcap_{i=1}^{n} X-U_{\alpha_i} = \Phi$.Therefore $\bigcup_{i=1}^{n} U_{\alpha_i}$ =X. Therefore X is

g*-compact.

Corollary 3.14: Let (X,τ) be a g^*-compact space and let $C_1 \supseteq C_2 \supseteq \ldots\ldots \supseteq C_n \supseteq C_{n+1} \supseteq \ldots\ldots$be a nested sequence of non-empty g^*-closed sets in X. Then $\bigcap_{n\in z^+} C_n$ is non-empty.

Proof: Obviously $\{C_n\}_{n\varepsilon\, z^+}$ has finite intersection property. Therefore by theorem 3.13 $\bigcap_{n\in z^+} C_n$is non-empty.

Theorem 3.15: Let *f* : (X,τ)→(Y,σ) be a function then,

(1) *f* is g^*-continuous, onto and X is g^*-compact $\Rightarrow$ Y is compact.

(2) *f* is continuous, onto and X is g^*-compact $\Rightarrow$ Y is compact.

(3) *f* is g^*-irresolute, onto and X is g^*-compact $\Rightarrow$ Y is g*-compact.

(4) *f* is strongly g^*-irresolute and X is g^*-compact $\Rightarrow$ Y is g*-compact.

(5) *f* is g^*-open, bijection and Y is g^*-compact $\Rightarrow$ X is compact.

(6) *f* is open, bijection and Y is g^*-compact $\Rightarrow$ X is compact.

(7) *f* is g^*-resolute, bijection and Y is g^*-compact $\Rightarrow$ X is g*-compact.

Proof of (1): Let $\{U_\alpha\}_{\alpha\varepsilon\Delta}$ be an open cover for Y. Then $\{ f^{1}(U_\alpha)\}_{\alpha\varepsilon\Delta}$ is a g*-open coverfor X.Since Xis g*-compact,there exists $\alpha_1,\alpha_2,\ldots\ldots\ldots\alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$.Therefore $Y = f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Therefore Y is compact.Proof for (2) to (7) are similar to the above.

4. G^*-COUNTABLY COMPACT SPACE

Definition 4.1: A subset A of a topological space (X,τ) is said to be g^*-countably compact if every countable g^*-open covering of A has a finite sub cover.

Example 4.2: An infinite cofinite topological space is g^*-countably compact.

Example 4.3: A countably infinite discrete topological space is not g^*-countably compact, because$\{\{x\}/x \in X\}$ is a countable g^*-open cover, which has no finite subcover.

Remark 4.4: Every g^*-compact space is g^*-countably compact.

Proof: It is obvious from definition.

Theorem 4.5: In a g^*-countably compact topological space every infinite subset has a g^*-limit point.

Proof: Let (X,τ) be g^*-countably compact. Suppose that there exists an infinite subsetwhich has no g^*-limit point. Let $B=\{a_n/ n \in N\}$ be a countable subset of A. Since B has no g^*-limit point, there exists a g^*-neigh bourhood U_n of $a_n$such that $B \cap U_n = \{a_n\}$.Now $\{U_n\}$ is a g^*-open cover for B. Since B^c is g^*-open $\{B^c, \{U_n\}_{n\in Z^+}\}$ is a countable g^*-open cover for X.But it has no finite sub cover which is a contradiction, since X is g^*-countably compact.Therefore every infinite subset of X has a g^*-limit point.

Corollary 4.6: In a g^*-compact topological space every infinite subset has a g^*-limit point.

Proof follows fromtheorem (4.5), since every g^*-compact space is g^*-countably compact.

Theorem 4.7: A g^*-closed subset of g^*-countably compact space is g^*-countably compact.

Proof is similar to theorem (3.5).

Definition 4.8: In a topological space (X,τ) a point $x \in X$ is said to be a g*-isolated pointof A if every g*-open set containing x contains no point of A other than x.

Theorem 4.9: LetX be a non empty g*-T_2 space. If X has no g*- isolated points then X is uncountable.

Proof: Let $x_1 \in X$. Choose a point y of X different from x.This is possible since $\{x_1\}$ is not a g*-isolated point. Since X is g*-T_2, there exists g*-open sets U_1 and V_1 such that $U_1 \cap V_1 = \Phi$; $x \in U_1$, $y \in V_1$. Therefore V_1 is g*-open and $x_1 \notin g^*cl(V_1)$. By repeating the sameprocess with V_1in the place of X and x_1 in the place of y we get a point $x \neq x_1$ and a g*-open set V_2 such that V_2 is g*-open and $x_2 \notin g^*cl(V_2)$. In general, given V_{n-1} which is g*-open and non empty, choose V_n to be a non empty g*-open set such that $V_n \subseteq V_{n-1}$ and $x_n \notin g^*cl(V_n)$.

Hence we get a nested sequence of g*-closed sets such that $g^*cl(V_n) \supseteq g^*cl(V_{n-1}) \supseteq \ldots\ldots\ldots\ldots$ Since X is $g^*compact$ $\cap$ $g^*cl(V_n) \neq \phi$. Therefore there exists $x \in \cap g^*cl(V_n)$. But $x \neq x_n$ for every n, since $x_n \notin g^*cl(V_n)$ and $x \in g^*cl(V_n)$. Define $f: Z \to X$ such that $f(n) = x_n$. Then $x \in X$ has no preimage.

Therefore f is not onto and hence X is uncountable.

Note: The converse of Theorem 4.5 is true in a g*-T_1 space.

Theorem 4.10: In a g*-T_1 space, if every infinite subset has a g*-limit point then X is g*-limit point then X is g*-countably compact.

Proof: Let every infinite subset has a g*-limit point. To prove X is g*-countably compact. If not there exists a countable g*-open cover $\{U_n\}$ such that it has no finite subcover. Since $U_1 \neq X$ there exists $x_1 \notin U_1$; Since $X \neq U_1 \cup U_2$ there exists $x_2 \notin U_1 \cup U_2$. Proceeding like this there exists $x_n \notin U_1 \cup U_2 \cup \ldots\ldots. \cup U_n$ for all n. $A = \{x_n\}$ is an infinite set. If $x \in X$ then $x \in U_n$ for some n. But $x_n \notin U_n$ for all k ≥ n. $U_n - \{x_1, x_2, \ldots\ldots x_{n-1}\}$ is a g*-open set (since X is g*-T_1) containing x which does not have a point of A other than x. Therefore x is not a limit point of A which is a contradiction.

Theorem 4.11: A topological space (X,τ) is g*-countably compact if and only if for every countable collection $\mathcal{C}$of g*-closed sets in X having finite intersection property, $\bigcap_{c \in C} C$ of all elements of $\mathcal{C}$ is non-empty.

Proof: Similar to the proof of Theorem (3.13).

Corollary 4.12: X is g*-countably compact if and only if every nested sequence of g*-closed non empty sets $C_1 \supset C_2 \supset \ldots\ldots\ldots$ has a non empty intersection.

Proof: Obviously $\{C_n\}_{n \in Z^+}$ has finite intersection property. Therefore by theorem (4.11) $\bigcap_{n \in z^+} C_n$ is non-empty.

5. SEQUENTIALLY G^*-COMPACT SPACE

Definition 5.1: A sequence $\{x_n\}$ in X is said to g^*-converge to a point x in X if for every g^*-open set U containing x, there exists a number N such that $x_n \in U$ $\forall n \geq N$ and we write $x_n \xrightarrow{g^*} x$.

Definition 5.2: A subset A of a topological space (X,τ) is said to be sequentially g*-compact, if every sequence in A contains a subsequence whichg*-converges to some point in A.

Example 5.3: Any finite topological space is sequentially g*-compact.

Example 5.4: An infinite indiscrete topological space is not sequentially g*-compact.

Theorem 5.5: A finite subset A of a topological space (X, τ) is sequentially g*-compact.

Proof: Let $\{x_n\}$ be an arbitrary sequence in X. Since A is finite, atleast one element of the sequence say x_o must be repeated infinite number of times.So the constant subsequence x_o, x_o……must g*-converges to x_o.

Remark 5.6: Sequentially g*-compactness implies sequentially compactness. Since open sets are g*-open. But the inverse implication is not true as seen in the following example.

Example 5.7: Any infinite indiscrete space is sequentially compact but not sequentially g*-compact.

Theorem 5.8: Every sequentially g*-compact space is g*-countably compact.

Proof: Let (X,τ) be sequentially g*-compact.Suppose X is not g*-countably compact.Then there exists countable g*-open cover $\{U_n\}_{n\in z^+}$ which has no finite subcover.Then X = $\bigcup_{n\in z^+} U_n$. Choose $x_1\in U_1, x_2\in U_2 - U_1, x_3\in U_3 - \bigcup_{i=1,2} U_i \ldots\ldots\ldots\ldots x_n\in U_n - \bigcup_{i=1}^{n-1} U_i$. This is possible since $\{U_n\}$ has no finite sub cover.Now $\{x_n\}$ is a sequence in X. Let $x\in X$ be arbitrary.Then $x\in U_k$ for some k. By our choice of $\{x_n\}$, $x_i \notin U_k$ for all *i* greater than *k*.Hence there is no subsequence of $\{x_n\}$which cang*-converge to *x*.Since *x* is arbitrary the sequence $\{x_n\}$ has no g*-convergent subsequence which is a contradiction. Therefore X is g*-countably compact.

Theorem 5.9: Let $f:(X,\tau)\to(Y,\sigma)$ be a bijection, then

(1) *f* is g*-resolute,bijection and Y is sequentially g*-compact ⇒ X is sequentially g*-compact.

(2) *f* is onto, g*-irresulote and X is sequentially g*-compact ⇒ Y is sequentially g*-compact.

(3) *f* is onto, continuous and X is sequentially g*-compact ⇒ Y is sequentially compact.

(4) *f* is onto, strongly g*-continuous and X is sequentially g*-compact ⇒ Y is sequentially g*-compact.

Proof of (1): Let$\{x_n\}$ be a sequence in X. Then $\{f(x_n)\}$is a sequence in Y.It has a g*-convergent subsequence $\{f(x_{n_k})\}$such that $f(x_{n_k})\xrightarrow{g^*} y_o$in Y. Then there exists $x_0\in X$ such that $f(x_0)=y_0$.Let U be a g*-open set containing x_0.Then $f(U)$ is a g*-open set containing y_0.Then there exists N such that $f(x_{n_k})\in f(U)$for all $k\geq N$. Therefore $f^{-1}of(x_{n_k})\in f^{-1}of(U)$. Therefore $x_{n_k}\in U$for all $k\geq N$.This proves that X is sequentially g*-compact. Proof for (2) to (5) is similar to the above.

6. G*-LOCALLY COMPACT SPACE

Definition 6.1: A topological space (X,τ) is said to be g*- locally compactif every point of*x*is contained in a

g^*-neighbourhoodwhose g*-closure is g*-compact.

Remark 6.2: Any g*-compact space is g^*- locally compact but the converse need not be true as seen in the following example.

Example 6.3: Let (X,τ) be an infinite indiscrete topological space.It is not g*-compact. But for every $x \in X, \{x\}$ is a g^*-neighbourhood and $\overline{\{x\}} = \{x\}$ is g*-compact. Therefore it is g^*- locally compact.

Theorem 6.4: Let (X, τ) be g*-multiplicative g*-T_2 space. Then *X* is g*-locally compact if and only if each of its points is a g*-interior point of some g*-compact subset of *X*.

Proof: Let *X* be g*-locally compact and $x \in X$. Then*x* has a g*-neighbourhood N such that g*cl(N) is g*-compact. Conversely, let every point $x \in X$ be a g*-interior point of some g*-compact subset of *X*. Given $x \in X$, there exists g*-compact subset N such that $x \in g^* \text{int}(N)$. So, N is a g*neighbourhood of *x*. By the hypothesis and theorem (3.8), N is g*-closed. Therefore *X* is g*-locally compact.

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