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RBF-BASED MESH-LESS METHOD FOR LARGE DEFLECTION OF THIN PLATES

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ABSTRACT

A simple, yet accurate, mesh-less method for the solution of thin plates undergoing large deflections is presented. The method is based on collocation with 5th order polynomial radial basis function. In order to address the in-plane edge conditions, two formulations, namely *w-F* and *u-v-w* are considered for the movable and immovable edge conditions, respectively. The resulted coupled nonlinear equations for the two cases are solved using an incremental-iterative procedure. The accuracy and efficiency of the method is verified through several numerical examples.

KEYWORDS: RBF, Mesh-Less, Plate, Movable, Immovable Edge

INTRODUCTION

In some applications of thin elastic plates, the deflections may increase under loading conditions beyond a certain limit recognized as large deformations. Because of these large deformations, the mid plane stretches and hence produces considerable in-plane stresses that are neglected by the small-deflection bending theory. For instance, in the case of circular plate with a clamped edge subject to uniform load and having a central deflection of 100% of its thickness, the maximum stretching stress is approximately 40% of the maximum bending stress [1]. For such situations, an extended plate theory must be employed, accounting for the effect of large deflections. Large elastic deflection of a thin elastic plate is governed by coupled non-linear differential equations, as discussed in the next section of the paper, for which analytical solutions are available only for very few cases involving simple geometries and loading conditions [1 through 5]. For other cases, the problem has to be solved using numerical techniques such as the finite difference methods (FDM), the finite element methods (FEM) and the boundary element methods (BEM).

Nevertheless, the possibility of obtaining numerical solutions without resorting to the mesh based techniques mentioned above, has been the goal of many researchers throughout the computational mechanics community for the past two decades or so. Radial Basis Function (RBF)-based collocation method, as one of the most recently developed numerical techniques, so-called mesh free or mesh less methods, has attracted attention in recent years especially in the area of computational mechanics [6,7, 8]. This method does not require mesh generation which makes them advantageous for 3-D problems as well as problems that require frequent re-meshing such as those arising in nonlinear analysis. Due to its simplicity to implement, it represents an attractive alternative to FDM, FEM and BEM as a solution method of nonlinear differential equations. However, it is only since rather recently that the RBFs have been used to approximate solutions to partial differential equations and therefore this area is still relatively unexplored.

The roots of RBF go back to the early 1970s, when it was used for fitting scattered data [9]. In 1982, Nardini and Brebbia coupled RBF with BEM in a technique called dual reciprocity-boundary element method to solve free vibration problems, where the RBF was used to transform the domain integrals into boundary integrals [10]. Thereafter, many researchers have used RBF in conjunction with BEM to solve various problems in computational mechanics.

The method however has not been applied directly to solve partial differential equations until 1990 by Kansa [11, 12]. Since then, many researchers have suggested several variations to the original method. In general, RBF-based collocation method expands the solution of a problem in terms of radial basis functions and chooses expansion coefficients such that the governing equations and boundary conditions are satisfied at some selected domain and boundary points. However, one of the important issues in applying this technique is the determination of the proper form of radial basis function for a given differential equation. Most of the available radial basis functions involve a parameter called shape factor which needs to be selected so that the required accuracy of the solution is attained. The selection of shape factors for different problems has been the subjective of many studies. In this paper, however, the simple 5th order polynomial RBF that does not involve a shape parameter is considered. The objective of this paper is to offer a simple yet accurate mesh-free method for the solution of thin elastic plates undergoing large deflection. The method is also applicable to other non-linear problems in various areas of computational mechanics.

GOVERNING EQUATIONS

w -F Formulation

The governing equations for large deflection of plates can be expressed in terms of the deflection *w* and a stress function *F* [1]:

$$\nabla^4 w = \frac{h}{D}\left[\frac{q}{h} + \left(\frac{\partial^2 F}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{\partial^2 F}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - 2\left(\frac{\partial^2 F}{\partial x \partial y}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)\right] \tag{1}$$

$$\nabla^4 F = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right] \tag{2}$$

Where q is the distributed load, *h* is the plate thickness and *D* is the flexural rigidity of the plate. The stress function *F* is related to the membrane forces N_x, N_y and N_{xy} by the following differential operators:

$$N_x = h\frac{\partial^2 F}{\partial y^2} \qquad N_y = h\frac{\partial^2 F}{\partial x^2} \qquad N_{xy} = -h\frac{\partial^2 F}{\partial x \partial y} \tag{3}$$

The bending moments M_x, M_y & M_{xy} are related to w by the following differential operators respectively:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \right) \tag{4-a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} + \right) \tag{4-b}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \tag{4-c}$$

The equivalent transverse shear forces V_x, V_y are given by:

$$V_x = Q_x - \frac{\partial M_{xy}}{\partial y}, V_y = Q_y - \frac{\partial M_{xy}}{\partial x}, \tag{5}$$

Where $Q_x = -D\frac{\partial}{\partial x}(\nabla\ w)$, $\quad Q_y = -D\frac{\partial}{\partial y}(\nabla\ w)$ (6)

The general boundary conditions for large deflection of plates can be classified into two types:

- Transverse boundary conditions which are encountered in both small and large deflection formulations. For this type, we will consider that at each boundary point there are two prescribed boundary conditions:

$$BC_{w1}(w)=0 \quad \text{Where} \quad BC_{w1}(w)=w \ \ or \ \ BC_{w1}(w)=V_n \tag{7-a}$$

$$BC_{w2}(w)=0 \quad \text{Where} \quad BC_{w2}(w)=\frac{\partial w}{\partial n} \ \ or \ \ BC_{w2}(w)=M_n \tag{7-b}$$

- In-plane boundary conditions which have to be addressed in the case of large deflection formulation. For this type, there are two possibilities:
  - **Movable Edge**

$$BC_{F1}(F)=BC_{F2}(F)=0 \quad \text{Where} \quad BC_{F1}(F)=F \,\&\, BC_{F2}(F)=\frac{\partial F}{\partial n} \tag{8-a}$$

The above two conditions are equivalent to zero in plane edge forces

  - **Immovable Edge**

$$u = v = 0 \tag{8-b}$$

It should be noted that the *w-F* formulation given above is readily applicable for only the movable edge case such that $F = \frac{\partial F}{\partial n} = 0$. However, in the case of immovable edge, it is extremely difficult to establish the boundary conditions in terms of the stress function and therefore, the $w-F$ formulation cannot be used directly. This fact might explain the rareness of the later case in the available literature on numerical solution for large deflection of plates. In the present study, the problem is overcome by deriving the governing equations in terms of the three displacemts components *u*, *v* and *w* as discussed in the following sub-section.

***u-v-w* Formulation**

The details of the derivation can be found in reference [13]. For briefest, we present only the final equations:

$$L_{11}(u) + L_{12}(\upsilon) = NL_1(w) \tag{9}$$

$$L_{12}(v) + L_{22}(u) = NL_2(w) \tag{10}$$

$$\nabla^4 w = \frac{q}{D} + NL_3(u,\upsilon,w) \tag{11}$$

Where,

$$L_{11} = \frac{2\partial_{xx} + (1-v)\partial_{yy}}{2(1-v^2)} \tag{12-a}$$

$$L_{12} = \frac{\partial_{xy}}{2(1-\nu)} \tag{12-b}$$

$$L_{22} = \frac{2\partial_{yy} + (1-\nu)\partial_{xx}}{2(1-\nu^2)} \tag{12-c}$$

$$NL_1(w) = -\frac{(1+\nu)w_{xy}w_y + w_x\left(2w_{xx} + (1-\nu)w_{yy}\right)}{2(1-\nu^2)}, \tag{12-d}$$

$$NL_2(w) = -\frac{(1+\nu)w_x w_{xy} + w_y\left((1-\nu)w_{xx} + 2w_{yy}\right)}{2(1-\nu^2)} \tag{12-e}$$

$$NL_3(u,\upsilon,w) = \frac{Ehw_{xy}}{D(1+\nu)}\left(u_y + \upsilon_x + w_x w_y\right) + \frac{Ehw_{xx}}{2D(1-\nu^2)}\left(2u_x + w_x^2 + \nu\left(2\upsilon_y + w_y^2\right)\right)$$

$$+\frac{Ehw_{yy}}{2D(1-\nu^2)}\left(2\upsilon_y + w_y^2 + \nu\left(2u_x + w_x^2\right)\right) \tag{12-f}$$

The form of transverse boundary conditions considered for this formulation is the same as that for the *w-F* formulation while the in-plane boundary condition is the one that corresponds to the immovable case, i.e. $u = v = 0$.

RBF for *w-F* formulation

Consider the 2-D computational domain (Figure 1) that represents the plate geometry. For collocation, we use node points distributed both along the boundary $\left(\underline{x}_B^j \ , j = 1,..., N_B\right)$, and over the interior $\left(\underline{x}_D^j, j = 1,...., N_D\right)$. Let $\underline{x}_p = \{\underline{x}_B , \underline{x}_D\}$, so that the total number of points called poles is $N_p = N_B + N_D$. The deflection, w, is interpolated linearly by suitable radial basis functions:

$$w(\underline{x}) = \sum_{j=1}^{N_D} \alpha_w^j \Phi\left(\left\|\underline{x} - \underline{x}_D^j\right\|\right) + \sum_{j=1}^{N_B} \beta_w^j BC_{w1}\left(\Phi\left(\left\|\underline{x} - \underline{x}_B^j\right\|\right)\right) + \sum_{j=1}^{N_B} \gamma_w^j BC_{w2}\left(\Phi\left(\left\|\underline{x} - \underline{x}_B^j\right\|\right)\right) \tag{13}$$

Similarly, for the stress function, *F*:

$$F(\underline{x}) = \sum_{j=1}^{N_D} \alpha_F^j \Phi\left(\left\|\underline{x} - \underline{x}_D^j\right\|\right) + \sum_{j=1}^{N_B} \beta_F^j BC_{F1}\left(\Phi\left(\left\|\underline{x} - \underline{x}_B^j\right\|\right)\right) + \sum_{j=1}^{N_B} \gamma_F^j BC_{F2}\left(\Phi\left(\left\|\underline{x} - \underline{x}_B^j\right\|\right)\right) \tag{14}$$

Where Φ is the the 5th order polynomial given by $\left\|\underline{x} - \underline{x}^j\right\|^5$. This radial basis function has the important advantage of not being dependent on a shape factor as the case for other radial basis functions. The $4N_B + 2N_D$ unknown coefficients: α_w^j, β_w^j, γ_w^j, α_F^j, β_F^j and γ_F^j can be determined by satisfying the governing equations at the N_D domain points, and by satisfying the corresponding boundary conditions at the N_B boundary points. The resulted equations can be expressed in the following matrix form:

$$\begin{bmatrix} BC_{w1}(\Phi) & BC_{w1}(BC_{w1}(\Phi)) & BC_{w1}(BC_{w2}(\Phi)) \\ BC_{w2}(\Phi) & BC_{w2}(BC_{w1}(\Phi)) & BC_{w2}(BC_{w2}(\Phi)) \\ \nabla^4\Phi & \nabla^4(BC_{w1}(\Phi)) & \nabla^4(BC_{w2}(\Phi)) \end{bmatrix} \cdot \begin{bmatrix} \alpha_w^i \\ \beta_w^i \\ \gamma_w^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{q}{D} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{h}{D} NL(w,F) \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \Phi & \Phi & \frac{\partial \Phi}{\partial n} \\ \frac{\partial \Phi}{\partial n} & \frac{\partial \Phi}{\partial n} & \frac{\partial}{\partial n}\left(\frac{\partial \Phi}{\partial n}\right) \\ \nabla^4\Phi & \nabla^4\Phi & \nabla^4\left(\frac{\partial \Phi}{\partial n}\right) \end{bmatrix} \cdot \begin{bmatrix} \alpha_F^i \\ \beta_F^i \\ \gamma_F^i \end{bmatrix} = E \begin{bmatrix} 0 \\ 0 \\ -\frac{E}{2} NL(w,w) \end{bmatrix} \tag{16}$$

Where $NL(w,F) = \frac{h}{D}\left(\left(\frac{\partial^2 F}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{\partial^2 F}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - 2\left(\frac{\partial^2 F}{\partial x \partial y}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)\right)$ and $NL(w,w)$ is obtained by replacing *F* by *w* in the foregoing expression.

In order to solve the above coupled and highly non-linear equations, the following incremental-iterative procedure is performed. In the following, the superscripts represent increments while subscripts represent iterations. As an example, the quantity $w_{i,xy}^k$ represents the second derivative of w with respect to x for the k^th increment and i^th iteration. This notation, however, does not apply to the coefficients α_w^j, β_w^j, etc.

- Apply the first load increment (k = 1) and obtain the solution of (15) and (16) by iterating the following steps:
  - Set the initial values of the second derivatives of w and F to zero, i.e.: $F_{0,xx}^1 = F_{0,yy}^1 = F_{0,xy}^1 = w_{0,xx}^1 = w_{0,yy}^1 = w_{0,xy}^1 = 0$ (In other words, the initial value of $NL\ (w_0^1, F_0^1)$ is equal to zero) and solve (15) for the coefficients $\alpha_w^j, \beta_w^j, \gamma_w^j$.
  - Use (13) to obtain the first estimate of deflection w_1^1 and calculate $NL\ (w_1^1, w_1^1)$. Note that w_1^1 corresponds to the solution of small deflection theory for the first increment.
  - Solve (16) for the for the coefficients α_F^j, β_F^j and γ_F^j then use them in (14) to obtain the first estimate of the stress function F_1^1
  - Update the right hand side of (14) by calculating $NL\ (w_1^1, F_1^1)$ and solve (15) for updated values of the coefficients $\alpha_w^j, \beta_w^j, \gamma_w^j$.
  - Use (13) to obtain the second estimate of deflection w_2^1 and calculate $NL\ (w_2^1, w_2^1)$
  - Repeat steps (c) to (e) until convergence is achieved.
- Use the values obtained for $NL\ (w_n^1, F_n^1)$ at the last iteration of the first load increment and add another load increment, then repeat steps (a) to (f).

- Continue adding increments until the total load is applied.

RBF for u-v-w Formulation

The deflection w is given by equation (13), while the in-plane deflections u and υ are given by:

$$u(\underline{x}) = \sum_{j=1}^{N_P} \alpha_u^j \Phi\left(\left\|\underline{x} - \underline{x}_P^j\right\|\right) \tag{17}$$

$$\upsilon(\underline{x}) = \sum_{j=1}^{N_P} \alpha_\upsilon^j \Phi\left(\left\|\underline{x} - \underline{x}_P^j\right\|\right) \tag{18}$$

Following the same procedure explained for the *w-F* formulation, we get the following two sets of coupled non-linear equations:

$$\begin{bmatrix} \Phi & 0 \\ 0 & \Phi \\ L_{11}(\Phi) & L_{12}(\Phi) \\ L_{12}(\Phi) & L_{22}(\Phi) \end{bmatrix} \cdot \begin{bmatrix} \alpha_u^j \\ \\ \\ \alpha_\upsilon^j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ NL_1(w) \\ NL_2(w) \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} BC_{w1}(\Phi) & BC_{w1}(BC_{w1}(\Phi)) & BC_{w1}(BC_{w2}(\Phi)) \\ BC_{w2}(\Phi) & BC_{w2}(BC_{w1}(\Phi)) & BC_{w2}(BC_{w2}(\Phi)) \\ \nabla^4\Phi & \nabla^4(BC_{w1}(\Phi)) & \nabla^4(BC_{w2}(\Phi)) \end{bmatrix} \cdot \begin{bmatrix} \alpha_w^i \\ \beta_w^i \\ \gamma_w^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{q}{D} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ NL_3(u,\upsilon,w) \end{bmatrix} \tag{20}$$

NUMERICAL EXAMPLES

The Following Numerical Examples Present the large deflection solution using RBF-based collocation method. The computer coding is done using the symbolic package *Mathematica* which enables finding the solutions for deflection and stresses as continuous functions of x and y. In all examples, the load is assumed to be uniformly distributed = q, Poisson ratio ν is assumed 0.3. For generality of the solutions, all results are made dimensionless, so that the coordinates, the load, the deflection and the stress are represented by $\overline{x} = x/a$, $\overline{y} = y/a$, $\overline{q} = qa^4/Eh^4$, $\overline{w} = w/h$ and $\overline{\sigma} = \sigma a^2/Eh^2$, respectively.

Example 1

A simply supported square plate ($w = M = 0$) with edges free to move boundary conditions $F = \frac{\partial F}{\partial n} = 0$. The plate is subjected to a uniformly distributed load $\overline{q}$ with a range of $2 \le \overline{q} \le 32$. The nodal distribution includes 36 boundary nodes and 81 domain nodes according to Figure 2. The maximum deflection at the centreline as obtained by the RBF method is compared to the one obtained by the finite element method software ANSYS. The two solutions are compared very well with a maximum error 0.86%. The membrane and bending stresses are compared in figure 1, respectively which shows again a very well agreement between the two solutions.

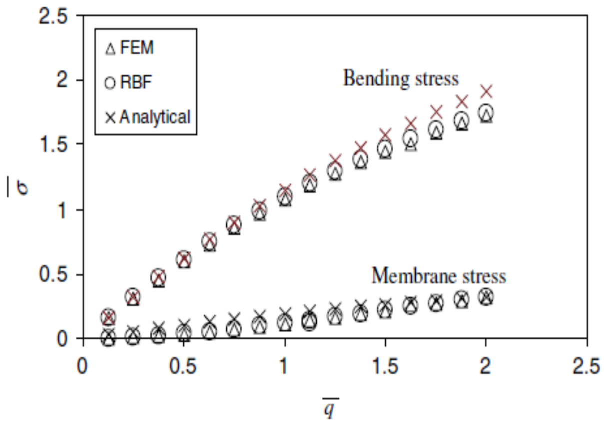

Figure 1: Stresses at the Centred for Simply Supported Circular Plate

Example 2

A clamped circular plate ($w = \frac{\partial w}{\partial n} = 0$) with edges free to move boundary conditions ($F = \frac{\partial F}{\partial n} = 0$) under uniformly distributed load $\bar{q}$ with a range of $0.5 \leq \bar{q} \leq 8.0$. The solution of this problem used 32 boundary nodes and 69 domain nodes. The deflection and stress solutions of the problem are given in figure 2-3. In figure 2, the analytical, FEM and RBF solutions for the maximum deflection at the centre of the plate are given. The figure shows that the three solutions are in good agreement. The results for the membrane and bending stresses are given in figure 3. The results in these figures show that while the two numerical solutions (RBF and FEM) are in good agreement both of them deviates from the analytical solution especially for higher loads. Observed deviations of the numerical solutions from the analytical solution can be attributed to the acknowledged inherent approximation of the analytical solution [1].

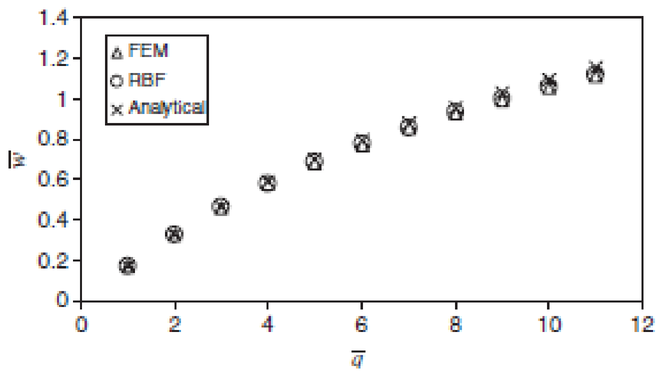

Figure 2: Central Deflection versus Load for Clamped Circular Plate

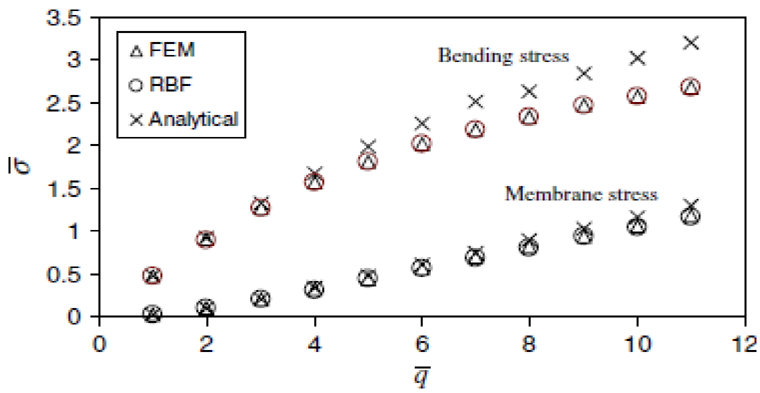

Figure 3: Stresses at the Centre for Clamped Circular Plate

Example 3

Consider a simply supported square plate subjected to a uniformly distributed load q which is increased from 2 to 32 with equal increments of 2. There is no analytical solution available for this problem and therefore the RBF solution is compared with FEM solution only.

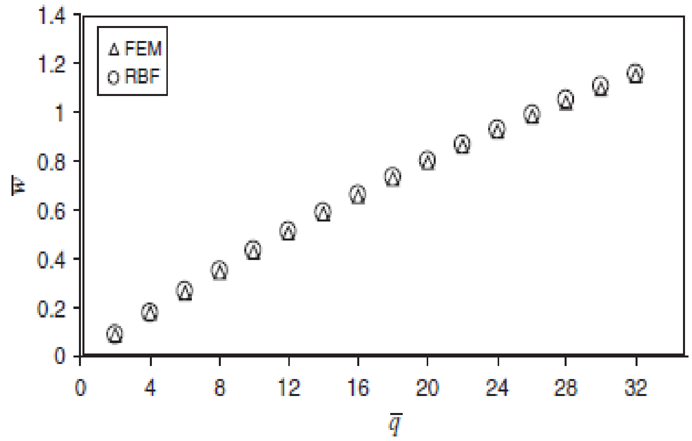

Figure 4: Central Deflection versus Load for Clamped Square Plate

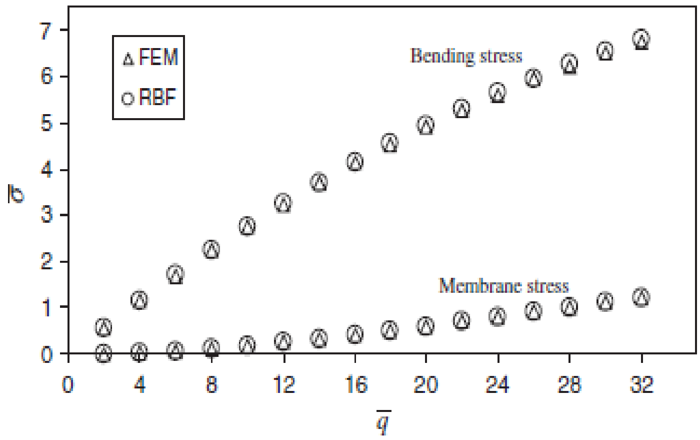

Figure 5: Stresses at the Center for Clamped Square Plate

CONCLUSIONS

A simple mesh-less method for the analysis of thin plates undergoing large deflections is presented. The method is based on collocations with the fifth order polynomial radial basis function. This radial basis function does not require a shape parameter that needs to be specified as the case for other well known radial basis functions. In addition, the method shares the same advantage of other RBF methods that do not require the computation of integrals or use of grids and meshes. The method also has the advantage of simplicity and has the important advantage of not being dependent on a shape factor as the case for other radial basis functions.

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